Multiple-Idler Parametric Amplifiers

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Abstract—**The input resistance and noise performance of multiple-idler parametric amplifiers are examined in this paper. General expressions applicable to any given amplifier are derived. Simplifications resulting from a sinusoidal elastance variation permit writing an expression for the input resistance of an amplifier having any given number of idlers by inspection.**

These expressions are then applied to examine the properties of an amplifier having two idlers. The conditions required for minimum noise performance are derived, and it is found that high pump frequencies and external resistive loading of one idler are required.

When the two-idler amplifier is compared to a conventional single-idler amplifier under those conditions which permit operation of the same diode at the same signal and pump frequencies, it is found that an improvement in noise figure results. However, the single idler amplifier pumped at the optimum frequency is capable of better noise performance, because minimum noise conditions cannot be satisfied for the two-idler device at this pump frequency. When below-signal-frequency pumping is utilized in the two-idler amplifier, the reduction in required pump power is substantial, but the noise figure is degraded by a minimum of approximately 3 dB.

INTRODUCTION

THE USE OF more than one idling frequency in a microwave varactor diode parametric amplifier results in a device which a preliminary examination shows to have significant desirable features.[1] Negative resistance amplification can be obtained with below signal frequency pumping and with low pump power. This is an advantage compared to a single-idler amplifier which requires a pump frequency well above the signal frequency in order to obtain low noise performance. Because the power required to pump a diode increases as the square of the pumping frequency, a much greater power is required for a single-idler amplifier.

The multiple-idler technique is not to be confused with the harmonic pumping technique. With harmonic pumping, a harmonoic of the fundamental reactance variation is used to generate the required idler frequency. This approach is unsatisfactory because the overall amplifier performance becomes noticeably degraded, since the variation of reactance at harmonic frequencies is very small compared to the fundamental component. In the multiple-idler technique, circuits are added at frequencies equal to the difference between harmonics of the pump frequency and the signal frequency. Any given idler frequency is directly generated by mixing of the pump with the signal or another idler. The generation of harmonics of the pump frequency as an intermediate operation is not necessary for multiple-idler amplification.

Because of limitations in the quality of varactor diodes, single idler amplifiers are limited in operation to Ku-band frequencies. While costly laboratory samples of extremely high quality diodes, i.e., diodes having cutoff frequencies in excess of 400 GHz, do exist, these diodes are difficult to fabricate. Therefore, when using readily available lower quality diodes, good low-noise performance is not possible at millimeter wavelength frequencies. Furthermore, even if adequate performance were available, excessively large pump power would be required at extremely high frequencies. This required pump power will most likely exceed the dissipation capability of the diode.

By using the multiple-idler technique with its associated lower pump power, good millimeter-wave performance might be obtained without the need for excessively high quality diodes. Although some theoretical analyses of multiple-idler parametric amplifiers have been published, none have come to any final conclusions using realistic circuit models about the relative merits of the different amplifier configurations.[1,2] To provide this information, this paper extends the analysis of single-idler parametric amplifiers to include the multiple-idler case, and then compares the performance in terms of low-noise receiver applications.

General equations for the input resistance and noise temperature of any small-signal, multiple-idler parametric amplifier are given in Section I, with the derivation of these equations included as Appendix I. These relations are then manipulated in Section II to the simpler forms resulting when a sinusoidal elastance variation exists; the proof of a theorem used in this section is given in Appendix II. Using these results, Section III analyzes the properties of the two-idler parametric amplifier. The gain and noise-figure characteristics are plotted. Comparison with the conventional single-idler amplifier is done in Section IV. Noise figure, input resistance, required pump power, and circuit complexities are compared. It is found that under certain conditions, when using the same diode at the same signal and pump frequencies, the two-idler amplifier has a lower noise figure, while with other conditions the single-idler device is better. While the two-idler device can give high gain performance with below-signal-frequency pumping and considerable savings in pump power, a significant sacrifice in noise figure must be made.

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[1] H. B. Henning, "New class of parametric amplifiers enables below signal pumping," *1963 Internat'l Conv. Rec.*, vol. II, pt. 3, pp. 90–97.

[2] V. I. Trifonov, "Multifrequency parametric amplifiers," *Radio Engrg. Electronic Phys.*, vol. 8, pp. 1365–1377, August 1963.

I. Basic Relationships

General equations for the properties of any multiple-idler parametric amplifier may be derived by extending the techniques of Penfield and Rafuse[3] in their evaluation of varactor diode circuit performance. The resulting equations are in a new form which is directly related to the diode and circuit parameters. However, because the derivation of these equations so closely follows the earlier work of Penfield and Rafuse, the derivation is given in Appendix I, and only the results used in the following sections are given here.

The input resistance of any multiple-idler parametric amplifier is given by

$$R_{\text{in}} = R_s + \text{Re}\left[Z_{s1}\frac{Z_{s1}^c}{Z_{ss}^c}\right] + \text{Re}\left[Z_{s2}\frac{Z_{s2}^c}{Z_{ss}^c}\right] + \cdots \tag{1}$$

where R_s is the series resistance of the varactor diode, and Z_{sn} and Z_{sn}^c are the elements and cofactors, respectively, of an impedance matrix whose elements are given by (34) and (39) of Appendix I. The external signal frequency impedance is not to be included in the evaluation of the input resistance.

The noise temperature of a multiple-idler parametric amplifier is found to be

$$T = -\frac{R_sT_d + \dfrac{|Z_{1s}^c|^2}{|Z_{ss}^c|^2}(R_sT_d + R_1T_1) + \dfrac{|Z_{2s}^c|^2}{|Z_{ss}^c|^2}(R_sT_d + R_2T_2) + \cdots}{R_s + \text{Re}\left[Z_{s1}\dfrac{Z_{s1}^c}{Z_{ss}^c}\right] + \text{Re}\left[Z_{s2}\dfrac{Z_{s2}^c}{Z_{ss}^c}\right] + \cdots}. \tag{2}$$

This equation does not show what is the minimum achievable noise temperature or how this minimum may be achieved. It is shown in Appendix I that this minimum is given by

$$T_{\min} = T_d\frac{2\omega_s}{m_t\omega_c}\left[\frac{\omega_s}{m_t\omega_c} + \sqrt{1 + \left(\frac{\omega_s}{m_t\omega_c}\right)^2}\right] \tag{3}$$

where m_t is the total modulation ratio of Penfield and Rafuse defined in (54) .This minimum noise temperature is possible if and only if the relation

$$r_ke^{j\theta_k} = r_k^1 \tag{4}$$

is satisfied for each and ever k, where r_k, r_k^1, and θ_k are defined by (50a–c). By using this condition, it becomes possible to determine the optimum pump frequency which produces the minimum of (2). However, under certain idler configurations, physically realizable conditions may not exist to achieve this minimum. Whether this minimum does in fact exist for a particular circuit can be determined only by seeing if (4) can be satisfied for that particular circuit.

The basic relationships given in this section are of a completely general nature and are applicable to all parametric amplifiers having any configuration of difference-frequency idlers. However, these relations do not provide a direct means of comparing different circuits and selecting the one best suited for a particular application. To compare the performance of particular circuits requires an independent evaluation of each circuit. In general, this can be a very tedious task. However, these relationships can still be used to provide great insight into the properties and limitations of multiple-idler parametric amplifiers. Some of the ways in which this is achieved is demonstrated by the remainder of this paper.

II. Properties of Amplifiers with Sinusoidal Elastance

In general, the manipulation of the matrices, evaluation of the determinants, and algebraic manipulations dictated by the equations derived in the previous section can be an overwhelming task and the resulting expressions far too unwieldy for practical application. However, by restricting the study to amplifiers in which the elastance varies sinusoidally with time, many simplifications can be made. Theoretically, this restriction limits the study to an ideal abrupt-junction varactor diode which is pumped with a sinusoidal current. However, in most practical situations, the elastance variation is close enough to being purely sinusoidal that trends which are observed in the restricted study can be used validly as an indication of the properties obtained in the general case.

In the case of a sinusoidal elastance, the loaded conversion matrix $\boldsymbol{Z}_L = \boldsymbol{Z}_c + \boldsymbol{Z}_{\text{term}}$ becomes

$$\boldsymbol{Z}_L = \begin{bmatrix} R_s + R_0 & -\dfrac{S_1}{j\omega_1} & 0 & 0 & \cdots \\ \dfrac{S_1^*}{j\omega_s} & R_s + R_1 & -\dfrac{S_1}{j\omega_2} & 0 & \cdots \\ 0 & \dfrac{S_1^*}{j\omega_1} & R_s + R_2 & -\dfrac{S_1}{j\omega_3} & \cdots \\ 0 & 0 & \dfrac{S_1^*}{j\omega_2} & R_s + R_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{5}$$

This expression results from the fact that in a current-pumped abrupt-junction diode, only the elastance coefficients

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[3] P. Penfield, Jr., and R. P. Rafuse, *Varactor Applications.* Cambridge, Mass.: M.I.T. Press, 1962.

S_0 and S_1 are nonzero.[4] In writing (5), it is assumed that each of the loops in the circuit is tuned to resonance, so that the resulting main diagonal terms become real quantities.

Operations with matrices such as that shown in (5) can be made comparatively simple by using the following theorem:

Theorem: The determinant of an $n \times n$ matrix of the form

$$M = \begin{bmatrix} a_{00} & a_{01} & 0 & 0 & 0 & 0 & \cdots \\ a_{10} & a_{11} & a_{12} & 0 & 0 & 0 & \cdots \\ 0 & a_{21} & a_{22} & a_{23} & 0 & 0 & \cdots \\ 0 & 0 & a_{32} & a_{33} & a_{34} & 0 & \cdots \\ 0 & 0 & 0 & a_{43} & a_{44} & a_{45} & \cdots \\ 0 & 0 & 0 & 0 & a_{45} & a_{55} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{6}$$

is given by

$$|M| = a_{00}a_{11}a_{22} \cdots a_{nm}B_{1n}B_{2n} \cdots B_{nn} \tag{7}$$

where the functions B_{mn} are defined by

$$B_{mn} = \left\{ 1 - \cfrac{Q_m}{1 - \cfrac{Q_{m+1}}{\begin{matrix}\cdot\\ \cdot\\ \cdot\end{matrix}\atop{1 - \cfrac{Q_{n-1}}{1 - Q_n}}}} \right\} \tag{8}$$

and the terms Q_k are loop quotients defined by

$$Q_k = \frac{a_{k-1,k}a_{k,k-1}}{a_{k-1,k-1}a_{kk}} \tag{9}$$

provided none of the main diagonal terms are zero. The proof of this theorem is given in Appendix II.

Input Resistance

Having the information given by this theorem, the characteristics of multiple-idler parametric amplifiers may be readily calculated. For example, the input resistance of an amplifier with sinusoidal elastance variation is found from (1) to be of the form

$$R_{\text{in}} = R_s + \operatorname{Re} Z_{s1} \frac{Z_{s1}^c}{Z_{ss}^c} \cdot \tag{10}$$

The various terms of this expression are as follows:

$$Z_{s1} = -\frac{S_1}{j\omega_1}$$

$$Z_{ss}^c = a_{11}a_{22}a_{33} \cdots a_{nn}B_{2n}B_{3n} \cdots B_{nn}$$

and

$$Z_{s1}^c = -a_{10}a_{22}a_{33} \cdots a_{nn}B_{3n} \cdots B_{nn}.$$

Z_{s1}^c is obtained by substituting a_{10} for a_{11} and zero for a_{21}. Therefore, Q_2 is zero and B_{2n} is unity. Using this data results in

$$\frac{Z_{s1}^c}{Z_{ss}^c} = -\frac{a_{10}}{a_{11}} \frac{1}{B_{2n}} \cdot$$

It is now necessary to realize that

$$Q_1 = -\frac{|S_1|^2}{\omega_s\omega_1 R_s(R_s + R_1)} = -\frac{m_1^2\omega_c^2}{\omega_s\zeta_1\omega_1}$$

$$Q_2 = -\frac{|S_1|^2}{\omega_1\omega_2(R_s + R_1)(R_s + R_2)} = -\frac{m_1^2\omega_c^2}{\zeta_1\omega_1\zeta_2\omega_2}$$

or, in general

$$Q_k = -\frac{|S_1|^2}{\omega_{k-1}\omega_k(R_s + R_{k-1})(R_s + R_k)} = -\frac{m_1^2\omega_c^2}{\zeta_{k-1}\omega_{k-1}\zeta_k\omega_k}$$

where

$$m_1\omega_c = \frac{|S_1|}{R_s} \tag{11}$$

is the varactor diode figure of merit defined by Penfield and Rafuse[5] and

$$\zeta_k = \frac{R_s + R_k}{R_s} \tag{12}$$

is a measure of the external resistive loading in the idler at frequency $k\omega_p - \omega_s$ by R_k.

Therefore, the input resistance of a parametric amplifier having n idlers is given by

$$R_{\text{in}} = R_s \left[1 - \cfrac{\cfrac{m_1^2\omega_c^2}{\omega_s\zeta_1\omega_1}}{1 + \cfrac{\cfrac{m_1^2\omega_c^2}{\zeta_1\omega_1\zeta_2\omega_2}}{1 + \cfrac{\cfrac{m_1^2\omega_c^2}{\zeta_2\omega_2\zeta_3\omega_3}}{\begin{matrix}\cdot\\ \cdot\\ \cdot\end{matrix}\atop{1 + \cfrac{m_1^2\omega_c^2}{\zeta_{n-1}\omega_{n-1}\zeta_n\omega_n}}}}} \right] . \tag{13}$$

Equation (13) can be examined to show some of the regions in the $\omega_p - \omega_s$ frequency plane where gain is impossible. Thus, for a single idler amplifier $R_{\text{in}} = 0$ whenever

$$\omega_s\omega_1 = m_1^2\omega_c^2$$

where for simplicity, $\zeta_1 = \zeta_2 = \cdots = \zeta_n = 1$ has been chosen. For a two-idler amplifier, $R_{\text{in}} = \infty$ whenever

$$\Omega_1\omega_2 = m_1^2\omega_c^2 \quad \cdot$$

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[4] *Ibid.*, ch. 7.

[5] *Ibid.*, pp. 81–87.

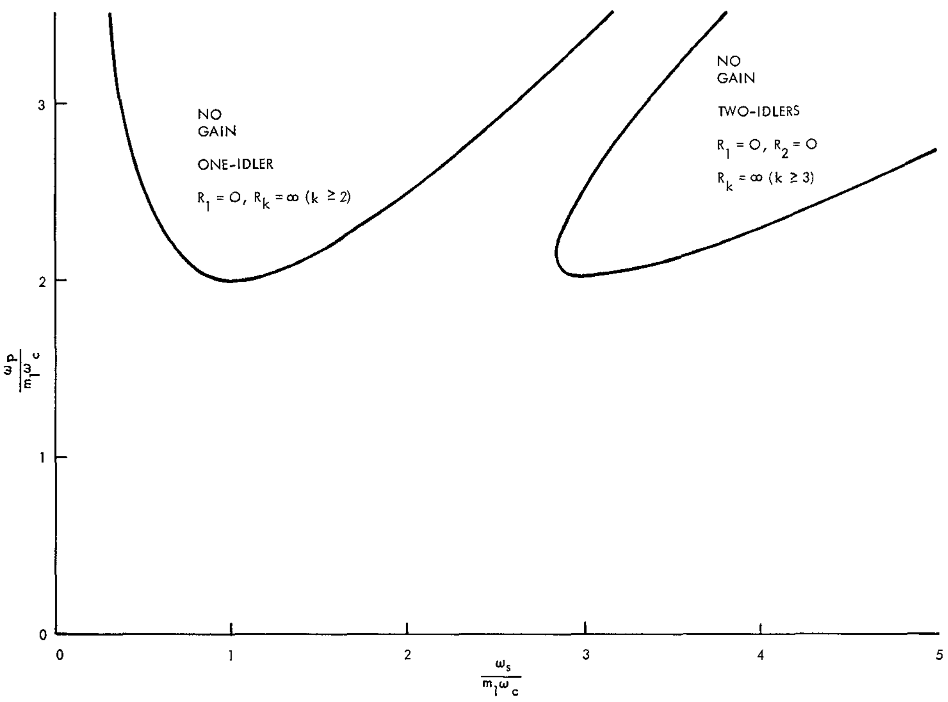

Fig. 1. Some regions of no gain for multiple-idler parametric amplifiers.

as can be seen from

$$R_{\text{in}} = R_s \left\{ 1 + \frac{\dfrac{m_1^2\omega_c^2}{\omega_s\Omega_1}}{1 - \dfrac{m_1^2\omega_c^2}{\Omega_1\omega_2}} \right\}$$

where $\Omega_1 = -\omega_1 = \omega_s - \omega_p$ is introduced so that for $\omega_s > \omega_p > (\omega_s/2)$ all quantities are positive. Because $R_{\text{in}} = 0$ and $R_{\text{in}} = \infty$ are boundaries between positive and negative input resistance, these are also boundaries between regions of possible gain and no gain, as illustrated in Fig. 1.

For a three-idler amplifier the input resistance is

$$R_{\text{in}} = R_s \left[1 + \cfrac{\dfrac{m_1^2\omega_c^2}{\omega_s\Omega_1}}{1 + \cfrac{\dfrac{m_1^2\omega_c^2}{\Omega_1\Omega_2}}{1 - \dfrac{m_1^2\omega_c^2}{\Omega_2\omega_3}}} \right].$$

When $\Omega_2\omega_2 = m_1^2\omega_c^2$, the input resistance is given by $R_{\text{in}} = R_s$. Since this is not a boundary of a region of possible gain, this is of no interest.

By continuing this process, it is seen that, in general, the relation

$$\Omega_{n-1}\omega_n = m_1^2\omega_c^2 (n > 2)$$

is not a boundary of possible gain. An earlier work[6] has shown that this is a boundary indicating when a negative resistance is converted into the $n-1$ idler in an n-idler amplifier. However, this condition does not satisfy the criterion that the entire amplifier must have a negative resistance as seen at the signal frequency. Therefore, the boundary in question does enclose a region over which gain is impossible, but it does not enclose the entire region.

III. TWO-IDLER PARAMETRIC AMPLIFIERS

The usefulness of the relationships derived in the previous sections will now be illustrated by using them to study in detail the properties of a parametric amplifier having two idlers, one at a frequency $\Omega_1 = -\omega_1 = \omega_s - \omega_p$ and the other at frequency $\omega_2 = 2\omega_p - \omega_s$.

Input Resistance and Gain Boundaries

By using (13), the input resistance of a two-idler parametric amplifier having a sinusoidal elastance is found to be given by

$$R_{\text{in}} = R_s \left\{ 1 + \frac{\dfrac{m_1^2\omega_c^2}{\omega_s\Omega_1}}{\zeta_1 - \dfrac{m_1^2\omega_c^2}{\Omega_1\omega_2}\dfrac{1}{\zeta_2}} \right\}. \tag{14}$$

The principal interest in this equation occurs for below signal-frequency pumping. Since in this case $(\omega_s/2) < \omega_p < \omega_s$, all terms in this expression are positive quantities. In order to have gain, it is necessary that the denominator term be negative; this can be achieved most readily by setting ζ_1 to a minimum value. The smallest value can be achieved by setting $R_1 = 0$, so that $\zeta_1 = 1$. The resulting expression for

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[6] Penfield and Rafuse,[3] pp. 185–190.

input resistance is given by

$$R_{\text{in}} = R_s \left\{ 1 + \frac{\dfrac{m_1{}^2\omega_c{}^2}{\omega_s \Omega_1}}{1 - \dfrac{m_1{}^2\omega_c{}^2}{\Omega_1 \omega_2} \dfrac{1}{\zeta_2}} \right\}. \tag{15}$$

To have gain, R_{in} must be negative. This is ensured by setting

$$1 + \frac{m_1{}^2\omega_c{}^2}{\zeta_2 \omega_1 \omega_2} < \frac{m_1{}^2\omega_c{}^2}{\omega_s \omega_1} \cdot \tag{16}$$

The boundary of this region, at which $R_{\text{in}} = 0$, is given by the equation

$$\omega_s \omega_1 \omega_2 \zeta_2 + m_1{}^2\omega_c{}^2(\omega_s - \omega_2 \zeta_2) = 0.$$

In terms of the pump frequency ω_p this boundary satisfies the relation

$$2\omega_s \zeta_2 \omega_1 - (3\zeta_2 \omega_s{}^2 + 2\zeta_2 m_1{}^2\omega_c{}^2)\omega_p + \omega_s \zeta_2(\omega_s{}^2 + m_1{}^2\omega_c{}^2) + \omega_s m_1{}^2\omega_c{}^2 = 0.$$

By solving for ω_p, the gain boundary for a two-idler parametric amplifier is found to be

$$\omega_p = \tfrac{3}{4}\omega_s + \frac{m_1{}^2\omega_c{}^2}{2\omega_s} \pm \tfrac{1}{2} \sqrt{\tfrac{1}{4}\omega_s{}^2 + m_1{}^2\omega_c{}^2\left(1 - \frac{2}{\zeta_2}\right) + \frac{m_1{}^4\omega_c{}^4}{\omega_s{}^2}} \cdot \tag{17}$$

There is an additional gain boundary, at which $R_{\text{in}} = \infty$, that can be obtained from (15). To have gain, the denominator,

$$1 - \frac{m_1{}^2\omega_c{}^2}{\Omega_1 \omega_2} \frac{1}{\zeta_2},$$

must be negative. This is satisfied by the condition

$$\Omega_1 \omega_2 \zeta_2 < m_1{}^2\omega_c{}^2. \tag{18}$$

Solving for the pump frequency ω_p results in the expression

$$\omega_p = \tfrac{3}{4}\omega_s \pm \tfrac{1}{2} \sqrt{\tfrac{1}{4}\omega_s{}^2 - 2\frac{m_1{}^2\omega_c{}^2}{\zeta_2}} \tag{19}$$

as the gain boundary. In the general case when $\zeta_1 \neq 1$, these boundaries are found to be given by

$$\omega_p = \tfrac{3}{4}\omega_s + \frac{m_1{}^2\omega_c{}^2}{2\zeta_1 \omega_s} \pm \tfrac{1}{2} \sqrt{\tfrac{1}{4}\omega_s{}^2 + \frac{m_1{}^2\omega_c{}^2}{\zeta_1}\left(1 - \frac{2}{\zeta_2}\right) + \frac{m_1{}^4\omega_c{}^4}{\zeta_1{}^2\omega_s{}^2}} \tag{17a}$$

and

$$\omega_p = \tfrac{3}{4}\omega_s \pm \tfrac{1}{2} \sqrt{\tfrac{1}{4}\omega_s{}^2 - 2\frac{m_1{}^2\omega_c{}^2}{\zeta_1 \zeta_2}} \tag{19a}$$

For the two extreme cases, $\zeta_2 = 1$, and $\zeta_2 = \infty$ (where the latter case corresponds to the one-idler parametric amplifier), the limits of possible gain are given by

$$\omega_p = \frac{\omega_s}{2} + \frac{m_1{}^2\omega_c{}^2}{\omega_s}; \qquad \omega_p = \omega_s$$

[from (17)]; and

$$\omega_p = \tfrac{3}{4}\omega_s \pm \tfrac{1}{2}\sqrt{\tfrac{1}{4}\omega_s{}^2 - 2m_1{}^2\omega_c{}^2} \quad \text{[from (19)]}$$

for $\zeta_2 = 1$ and by

$$\omega_p = \omega_s + \frac{m_1{}^2\omega_c{}^2}{\omega_s}; \qquad \omega_p = \frac{\omega_s}{2} \quad \text{[from (17)]};$$

and

$$\omega_p = \omega_s; \qquad \omega_p = \frac{\omega_s}{2} \quad \text{[from (19)]};$$

for $\zeta_2 = \infty$. These results are plotted later in this section.

Noise Temperature

By direct application of (2), the noise temperature of a two-idler parametric amplifier is found to be given by

$$T = \frac{\zeta_1 \zeta_2 + \dfrac{m_1{}^2\omega_c{}^2}{\omega_1 \omega_2} + \dfrac{m_1{}^2\omega_c{}^2}{\omega_1{}^2} \dfrac{\left(\zeta_2{}^2 + \dfrac{m_1{}^2\omega_c{}^2}{\omega_2{}^2}\right)}{\left(\zeta_1 \zeta_2 + \dfrac{m_1{}^2\omega_c{}^2}{\omega_1 \omega_2}\right)}}{\zeta_2 \dfrac{m_1{}^2\omega_c{}^2}{\omega_s \omega_1} - \zeta_1 \zeta_2 - \dfrac{m_1{}^2\omega_c{}^2}{\omega_1 \omega_2}} T_d + \frac{\left(\dfrac{m_1{}^2\omega_c{}^2}{\omega_1{}^2} \zeta_2{}^2\right)\dfrac{R_1}{R_s} T_1 + \left(\dfrac{m_1{}^2\omega_c{}^2}{\omega_1{}^2} \dfrac{m_1{}^2\omega_c{}^2}{\omega_2{}^2}\right)\dfrac{R_2}{R_s} T_2}{\left(\zeta_1 \zeta_2 + \dfrac{m_1{}^2\omega_c{}^2}{\omega_1 \omega_2}\right)\left(\zeta_2 \dfrac{m_1{}^2\omega_c{}^2}{\omega_s \omega_1} - \zeta_1 \zeta_2 - \dfrac{m_1{}^2\omega_c{}^2}{\omega_1 \omega_2}\right)} \cdot \tag{20}$$

It is of interest to determine if it is possible to achieve the minimum noise temperature predicted by (53), namely,

$$T_{\min} = T_d \frac{2\omega_s}{m_1 \omega_c} \left[\frac{\omega_s}{m_1 \omega_c} + \sqrt{1 + \left(\frac{\omega_s}{m_1 \omega_c}\right)^2} \right]. \tag{21}$$

The conditions necessary to achieve this minimum are found from the relation $r_1 e^{i\theta_1} = r_1{}'$ in accordance with (4). To simplify this investigation, the noise contributions from the external idler loads are eliminated by refrigerating these idler terminations so that T_1 and T_2 become absolute zero. It is found that the amplifier must be pumped at a frequency given by

$$\omega_p = \frac{1}{2}\,\frac{\omega_s^2 + m_1^2\omega_c^2}{\zeta_1} + \frac{(3\zeta_1 - 2)}{4\zeta_1}\,\omega_s + \frac{1}{4\zeta_1}\sqrt{[2\sqrt{\omega_s^2 + m_1^2\omega_c^2} + (\zeta_1 - 2) - \omega_s] - 8\,\frac{\zeta_1}{\zeta_2}\,m_1^2\omega_c^2}. \tag{22}$$

To illustrate the results obtained in this section so far, plots of the optimum pump frequency together with plots of the gain boundaries for particular two-idler parametric amplifiers are shown in Figs. 2–4 for the cases $\zeta_2 = 1$, $\zeta_2 = 2$, and $\zeta_2 = \infty$, respectively.

When operating at the optimum pump frequency, the noise performance of a two-idler amplifier is equal to the performance of the single-idler amplifier. However, for those cases in which gain is possible with below signal frequency pumping, i.e., with $\omega_s > \omega_p$, the optimum frequency is not defined below a comparatively high signal frequency. Since low-noise performance is obtained when the signal frequency is less than $m_1\omega_c$, it can be concluded that a single-idler parametric amplifier under optimum conditions results in noise performance which is superior to that obtainable from a two-idler parametric amplifier. The noise performance limits will be compared later.

Minimum Noise Conditions for Low $\omega_s/m_1\omega_c$

So far it has been shown that even under the idealized conditions of idler terminations cooled to absolute zero, the two-idler parametric amplifier usually results in a noise temperature exceeding that available from an optimum single-idler device. However, the value of the lowest achievable noise temperature still needs to be determined. From (20) the noise temperature, assuming $T_d = T_1 = T_2$, is found to be

$$T = \frac{\zeta_1\zeta_2 + \dfrac{m_1^2\omega_c^2}{\omega_1\omega_2} + \dfrac{m_1^2\omega_c^2}{\omega_1^2}\;\dfrac{\zeta_1\zeta_2^2 + \zeta_2\dfrac{m_1^2\omega_c^2}{\omega_2^2}}{\zeta_1\zeta_2 + \dfrac{m_1^2\omega_c^2}{\omega_1\omega_2}}}{\zeta_2\dfrac{m_1^2\omega_c^2}{\omega_s\omega_1} - \zeta_1\zeta_2 - \dfrac{m_1^2\omega_c^2}{\omega_1\omega_2}}\,T_d. \tag{23}$$

For a given signal frequency, this function depends upon ω_p, ζ_1, and ζ_2. The value of ζ_2 which minimizes T may be found by setting $\partial T/\partial\zeta_2 = 0$ and solving for ζ_2. The result of this manipulation is

$$\zeta_{2,\mathrm{opt}} = \frac{m_1^2\omega_c^2}{\omega_1\omega_2}\;\frac{\left[\zeta_1\left(\dfrac{m_1\omega_c}{\omega_s} + \dfrac{m_1\omega_c}{\omega_1}\right) + \zeta_1\left(\dfrac{m_1\omega_c}{\omega_1} - \dfrac{m_1\omega_c}{\omega_2}\right)\left[\dfrac{m_1^3\omega_c^3}{\omega_s^2\omega_1} + \dfrac{m_1^3\omega_c^3}{\omega_s\omega_1\omega_2} + \zeta_1\left(\dfrac{m_1\omega_c}{\omega_1} - \dfrac{m_1\omega_c}{\omega_2}\right)\right]\right]}{\zeta_1\dfrac{m_1^2\omega_c^2}{\omega_s\omega_1}\left(\dfrac{m_1\omega_c}{\omega_1} - \dfrac{m_1\omega_c}{\omega_2}\right) - \zeta_1^2\left(\dfrac{m_1\omega_c}{\omega_s} + 2\,\dfrac{m_1\omega_c}{\omega_1} - \dfrac{m_1\omega_c}{\omega_2}\right)}.$$

This lengthy equation is expressed as a function of the three variables ω_s, ω_1, and ω_2. However, since these are determined by only two independent variables, namely ω_s and ω_p, one of these variables can be eliminated. By expressing ω_1 as a

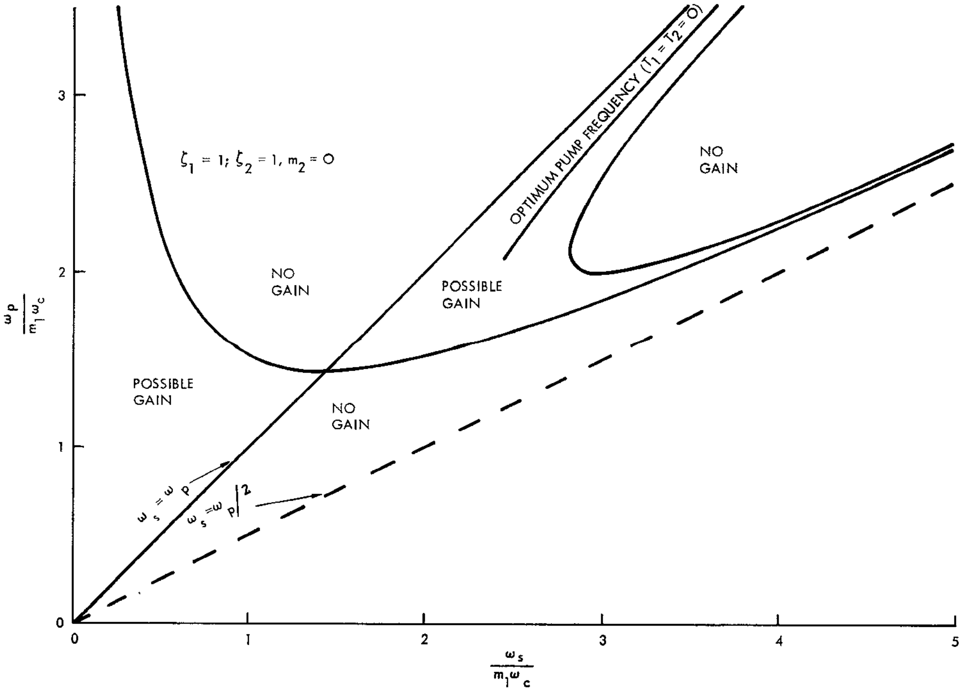

Fig. 2. Optimum pump frequency and regions of possible gain for a two-idler parametric amplifier ($\zeta_1 = 1$, $\zeta_2 = 1$, $m_2 = 0$).

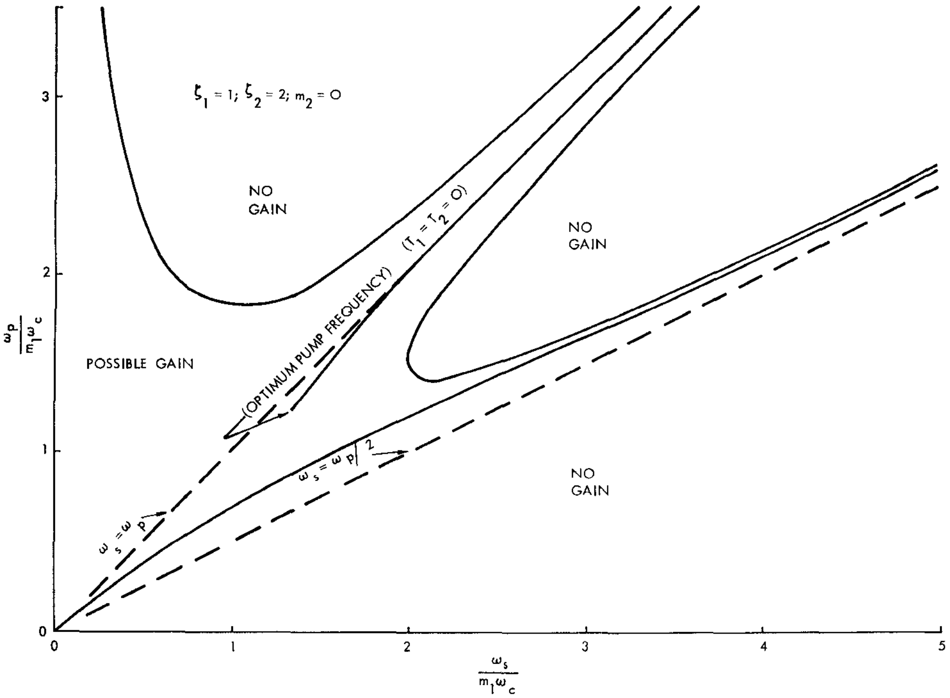

Fig. 3. Optimum pump frequency and regions of possible gain for a two-idler parametric amplifier ($\zeta_1=1$, $\zeta_2=2$, $m_2=0$).

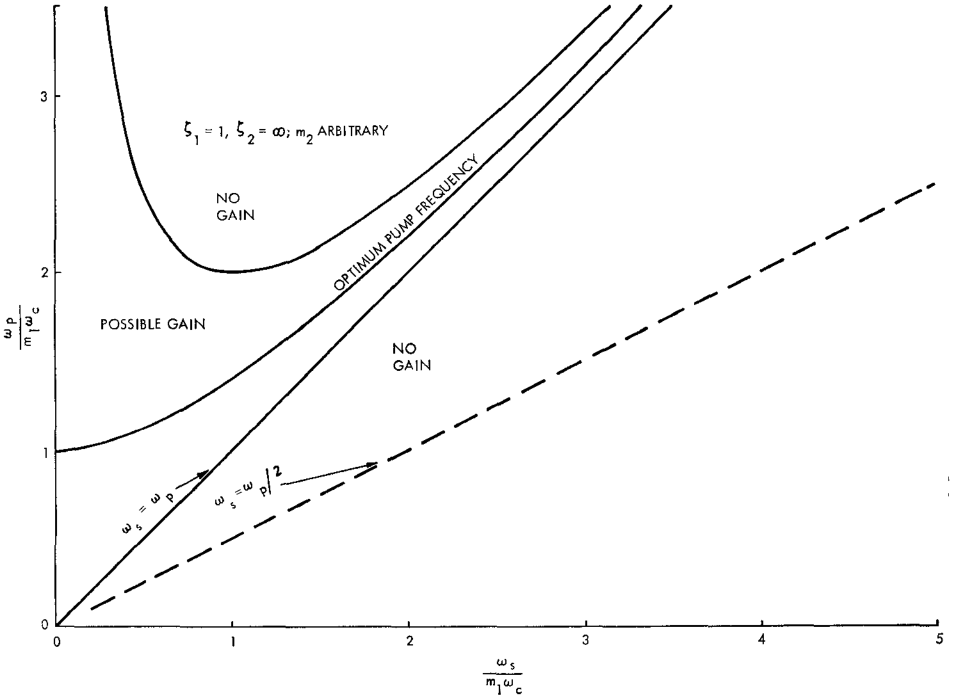

Fig. 4. Optimum pump frequency and regions of possible gain for a two-idler parametric amplifier ($\zeta_1=1$, $\zeta_2=\infty$, m_2=arbitrary).

function of ω_2 and ω_s, the above equation reduces to

$$\zeta_{2,\text{opt}} = \frac{2}{\dfrac{\zeta_1\omega_2}{m_1\omega_c}\left[\sqrt{\dfrac{2}{\zeta_1} + \left(\dfrac{\omega_s}{m_1\omega_c}\right)^2} - \dfrac{\omega_2}{m_1\omega_c}\right]}. \qquad (24)$$

Because ζ_2 must always be positive, the pump frequency must satisfy the condition

$$\frac{\omega_s}{2m_1\omega_c} < \frac{\omega_p}{m_1\omega_c} < \frac{\omega_s}{2m_1\omega_c}\left[1 + \sqrt{\frac{2}{\zeta_1}\left(\frac{m_1\omega_c}{\omega_s}\right)^2 + 1}\right].$$

For large values of $m_1\omega_c/\omega_s$, $\zeta_{2,\text{oqt}}$ approaches the value

$$\zeta_{2,\text{opt}} = \left[\frac{\sqrt{\dfrac{2}{\zeta_1}}}{2\dfrac{\omega_p}{\omega_s} - 1}\right]\frac{m_1\omega_c}{\omega_s}.$$

By setting $\partial\zeta_{2,opt}/\partial\omega_2 = 0$, it is found that $\zeta_{2,opt,\text{min}}$ is given by

$$\zeta_{2,\text{opt,min}} = \frac{8}{2 + \zeta_1\left(\dfrac{\omega_s}{m_1\omega_c}\right)^2},$$

which occurs when

$$\frac{\omega_p}{m_1\omega_c} = \frac{\omega_s}{2m_1\omega_c}\left[1 + \frac{1}{2}\sqrt{\frac{2}{\zeta_1}\left(\frac{m_1\omega_c}{\omega_s}\right)^2 + 1}\right].$$

Knowing $\zeta_{2,\text{opt}}$, the minimum noise figure can be calculated from

$$F_{\text{min}} = \left[1 + \frac{T}{T_0}\right]_{\text{min}}. \qquad (25)$$

By substituting (23) and (24) into (25), and by eliminating ω_1 as before, this minimum noise figure is found to be

$$F_{\text{min}} = \left[\frac{1}{1 - \dfrac{\omega_s}{2\omega_p}}\right]\left[1 + \zeta_1\left(\frac{\omega_s}{m_1\omega_c}\right)^2 + \zeta_1\frac{\omega_s}{m_1\omega_c}\sqrt{\frac{2}{\zeta_1} + \left(\frac{\omega_s}{m_1\omega_c}\right)^2}\right]. \qquad (26)$$

By inspecting this equation, it is seen that F_{min} is lowest when ω_p is maximized and ζ_1 is minimized. Hence setting $\zeta_1 = 1$ results in lowest noise figure and greatest negative input resistance.

From (26), the overall noise figure in dB can be expressed as the sum of two functions, one of which depends on the ratio ω_p/ω_s, and the other which depends on the ratio $\omega_s/m_1\omega_c$. In other words,

$$F_{\text{min,dB}} = F_{1,\text{dB}} + F_{2,\text{dB}} \qquad (27)$$

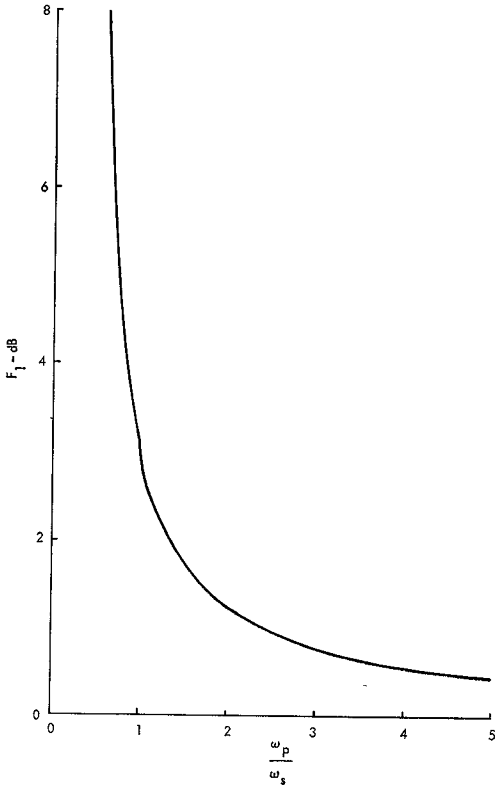

Fig. 5. Noise figure of a two-idler parametric amplifier using a lossless nonlinear capacitor.

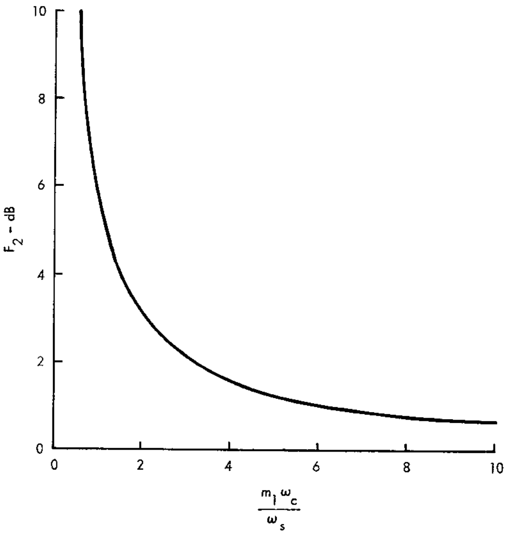

Fig. 6. Noise figure degradation caused by a lossy varactor in a two-idler parametric amplifier.

and

$$F_1 = \frac{1}{1 - \dfrac{\omega_s}{2\omega_p}} \qquad (28)$$

and

$$F_2 = 1 + \left(\frac{\omega_s}{m_1\omega_c}\right)^2 + \frac{\omega_s}{m_1\omega_c}\sqrt{2 + \left(\frac{\omega_s}{m_1\omega_c}\right)^2}. \qquad (29)$$

F_1 represents the noise figure of a two-idler parametric amplifier using a lossless nonlinear capacitor; its value can also be calculated by properly using the Manley-Rowe general energy relations.[7] F_2 indicates the degradation in noise figure caused by the loss mechanism associated with R_s. These two noise figure components are shown in Figs. 5 and 6.

IV. COMPARISON OF ONE-IDLER AND TWO-IDLER PARAMETRIC AMPLIFIERS

It has been shown in this paper that gain is possible for a two-idler parametric amplifier using a lossy varactor diode. However, it remains to be determined if the negative input impedance is of the same order of magnitude as that presently obtained from the single-idler amplifier; this, in turn, determines if comparable signal frequency circuits can be used. By substituting the low-noise value of ζ_2 into (13), the input resistance of a two-idler amplifier is found to be

$$R_{\text{in}} = R_s\left[1 - \frac{2}{\dfrac{\omega_s}{m_1\omega_c}\sqrt{2 + \left(\dfrac{\omega_s}{m_1\omega_c}\right)^2} - \dfrac{\omega_s^2}{m_1^2\omega_c^2}}\right]. \quad (30)$$

For large values of $m_1\omega_c/\omega_s$, R_{in} is given by

$$R_{\text{in}} \approx R_s\left(1 - \sqrt{2}\,\frac{m_1\omega_c}{\omega_s}\right).$$

The input resistance of the one-idler amplifier is found from (13) to be

$$R_{\text{in}} = R_s\left(1 - \frac{m_1^2\omega_c^2}{\omega_s\zeta_1\omega_1}\right).$$

When the low noise conditions

$$\zeta_1 = 1$$

and

$$\omega_p = \sqrt{\omega_s^2 + m_1^2\omega_c^2}$$

are inserted, this expression becomes

$$R_{\text{in}} = R_s\left[1 - \frac{1}{\dfrac{\omega_s}{m_1\omega_c}\sqrt{1 + \left(\dfrac{\omega_s^2}{m_1\omega_c}\right)} - \dfrac{\omega_s^2}{m_1^2\omega_c^2}}\right]; \quad (31)$$

for large values of $m_1\omega_c/\omega_s$, R_{in} is approximated by

$$R_{\text{in}} = R_s\left(1 - \frac{m_1\omega_c}{\omega_s}\right).$$

The input resistances of the two amplifiers are compared in Fig. 7. It is seen that they are of the same order of magnitude.

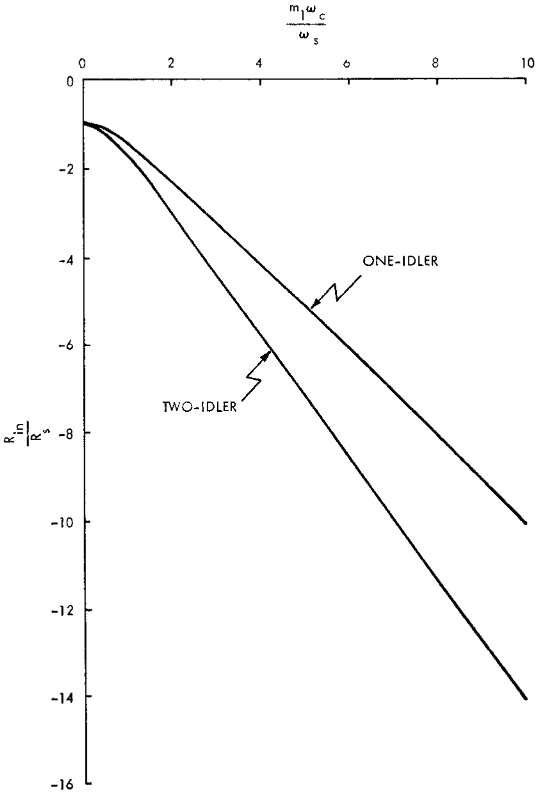

Fig. 7. Parametric amplifier input resistance when tuned for minimum noise figure.

Perhaps the most striking feature of the two-idler parametric amplifier when compared to the single-idler amplifier is its lower pump power requirements. This results from the fact that the power required for full pumping increases as the square of the pumping frequency.[8] Thus, for example, a fully pumped current pumped abrupt junction varactor diode requires a pumping power of

$$P = 0.500P_{\text{norm}}\left(\frac{\omega_p}{\omega_c}\right)^2$$

where P_{norm} is the normalization power defined as

$$P_{\text{norm}} = \frac{(\phi - V_B)^2}{R_s}.$$

Therefore, since the two-idler amplifier permits the use of much lower pump frequencies, the required pump power can be drastically reduced.

The noise temperature of a single-idler amplifier, derived from (2), is

$$T = \frac{\left(1 + \dfrac{m_1^2\omega_c^2}{\zeta_1^2\omega_1^2}\right)R_sT_d + \dfrac{m_1^2\omega_c^2}{\zeta_1^2\omega_1^2}R_1T_1}{\left[\dfrac{m_1^2\omega_c^2}{\zeta_1\omega_s\omega_1} - 1\right]}.$$

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[7] J. M. Manley and H. E. Rowe, "Some general properties of nonlinear elements—part I. General energy relations," *Proc. IRE*, vol. 44, pp. 904–913, July 1956.

[8] Penfield and Rafuse,[3] ch. 7.

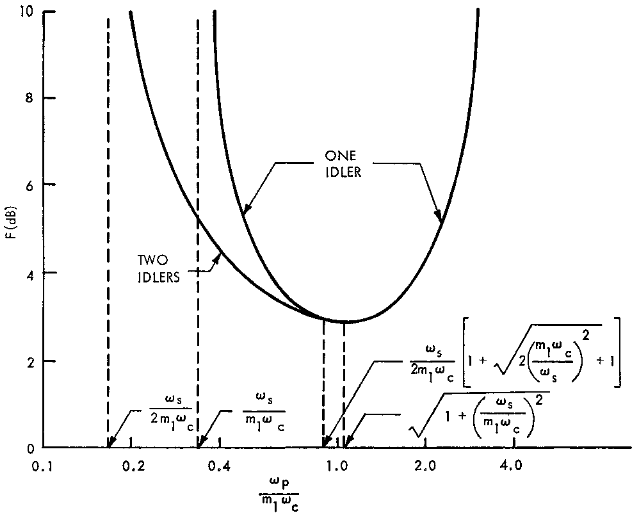

Fig. 8. Minimum noise figure of single-idler and two-idler parametric amplifiers (values calculated for $m_1\omega_c/\omega_s = 3$).

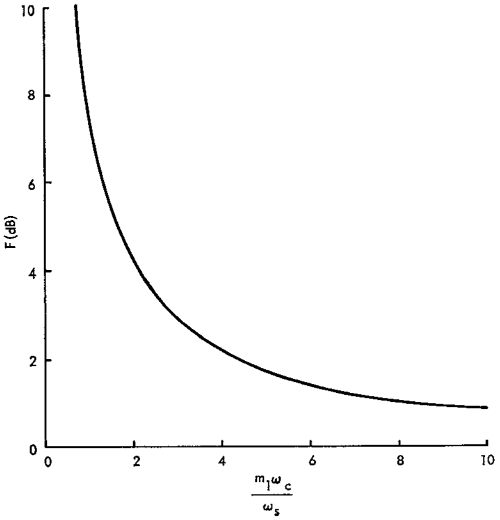

Fig. 9. Noise figure of a single-idler parametric amplifier.

Assuming, as before, $T_d = T_1$, this is equivalent to a room temperature noise figure of

$$F = \left(\frac{1}{1 - \frac{\omega_s}{\omega_p}}\right)\left(\frac{1}{1 - \frac{\zeta_1\omega_s\omega_1}{m_1^2\omega_c^2}}\right).$$

The expression within the first set of parentheses decreases with increasing pump frequency, while the second expression increases. The minimum noise temperature of (21) is achieved when $\zeta_1 = 1$ and[9]

$$\omega_p = \sqrt{\omega_s^2 + (m_1\omega_c)^2}.$$

The variation of noise figure with pump frequency for both types of amplifiers is found to occur as illustrated in Fig. 8. It can be seen that for those pump frequencies when the minimum noise figure of both amplifiers are simultaneously defined, i.e., for

$$\frac{\omega_s}{m_1\omega_c} < \frac{\omega_p}{m_1\omega_c} \leq \frac{\omega_s}{2m_1\omega_c}\left[1 + \sqrt{2\left(\frac{m_1\omega_c}{\omega_s}\right)^2 + 1}\right],$$

the two-idler device results in a lower noise figure. However, the lowest obtainable noise figure of the single idler circuit, which occurs at

$$\frac{\omega_p}{m_1\omega_c} = \sqrt{1 + \left(\frac{\omega_s}{m_1\omega_c}\right)^2},$$

is lower than any noise figure obtainable with the two-idler amplifier.

If the two-idler amplifier is pumped below the signal frequency, its noise figure increases to a level well in excess of the values normally obtained with the conventional single-idler amplifier. The minimum noise figure of the single-idler amplifier is given in Fig. 9. This illustration is almost identical to the noise-figure degradation in a two-idler amplifier shown in Fig. 6. Therefore, referring to Fig. 5, it is seen that when the two-idler amplifier is operated below the signal frequency, the total noise figure is worse than that of the single-idler circuit by at least approximately 3 dB.

One obvious drawback of the two-idler amplifier is that more complicated circuitry is required.[10] This results from the fact that it is necessary to provide independent tuning at four different frequencies.

V. CONCLUSIONS

The properties of multiple-idler parametric amplifiers may be determined from equations derived by general matrix techniques. Although for the general case, the use of these equations becomes excessively tedious, many simplifications result by assuming that the elastance variation is sinusoidal; in fact, an expression for the input resistance may be written by inspection for any number of idlers. Multiple-idler amplifiers permit the use of a low pump frequency, even lower than the signal frequency. As a result, much lower pump power is required. However, as the number of idlers increases, the complexity of the circuit becomes overwhelming.

When comparing a two-idler amplifier with a one-idler amplifier, it is found that comparable gain can be obtained with a great savings in pump power, by using a more complicated circuit. However, when using low pump frequencies, the noise figure becomes excessively high.

When both amplifiers are pumped the same way, the two-idler circuit has a potentially lower noise figure under certain conditions.

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[9] Penfield and Rafuse,[3] pp. 175–178.

[10] See, for example, J. Vilcans and J. Ginsberg, "35-Gc parametric amplifier with 29-Gc pump," *NEREM Record*, vol. 6, pp. 142–143, 1964.

APPENDIX I

DERIVATION OF SOME BASIC RELATIONSHIPS

The techniques which are used in this study are extensions of those methods introduced by Penfield and Rafuse in their evaluation of varactor diode circuit performance. Thus, the varactor diode is represented by an equivalent circuit consisting of a series combination of a constant resistance R_s and a time-varying elastance $S(t)$, as shown in Fig. 10. Under small signal conditions in which the elastance variation is essentially determined by only a single source designated as the pump, the voltage across the varactor diode is given by

$$v(t) = S(t)\int i(t)dt + R_s i(t) + e_n. \tag{32}$$

where e_n is the noise voltage generated within R_s. By using Fourier transform techniques Penfield and Rafuse show that (32) may be expressed in matrix form as[11]

$$V = Z_c I + E_n, \tag{33}$$

where each of the matrix terms is given by

$$V = \begin{bmatrix} V_s \\ V_1{}^* \\ \cdot \\ \cdot \\ V_k{}^* \\ \cdot \\ \cdot \end{bmatrix}, \quad I = \begin{bmatrix} I_s \\ I_1{}^* \\ \cdot \\ \cdot \\ I_k{}^* \\ \cdot \\ \cdot \end{bmatrix}, \quad E_n = \begin{bmatrix} E_{ns} \\ E_{n1}{}^* \\ \cdot \\ \cdot \\ E_{nk}{}^* \\ \cdot \\ \cdot \end{bmatrix} \tag{34a, b, c}$$

and

$$Z_c = \begin{bmatrix} R_s + \dfrac{S_0}{j\omega_s} & -\dfrac{S_1}{j\omega_1} & -\dfrac{S_2}{j\omega_2} & \cdots \\ \dfrac{S_1{}^*}{j\omega_s} & R_s - \dfrac{S_0}{j\omega_1} & -\dfrac{S_1}{j\omega_2} & \cdots \\ \dfrac{S_2{}^*}{j\omega_s} & -\dfrac{S_1{}^*}{j\omega_1} & R_s - \dfrac{S_0}{j\omega_2} & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{34d}$$

and the kth element is the corresponding component at frequency $k\omega_p - \omega_s$. In this paper, these matrices are restricted to only the difference frequency components, because these are the only ones which contribute to the amplification mechanism.[12]

In a general multiple-idler parameteric amplifier, a varactor diode is incorporated into a circuit such as that shown in Fig. 11. Each idler termination at frequency ω_k is an impedance $Z_{Tk} = R_k + jX_k$ with an associated noise voltage E_{nTk}. The signal voltage across the diode will be of the form

$$V_s = Z_{\text{in}} I_s + E_{nt} \tag{35}$$

where Z_{in} and E_{nt} depend upon all the idler circuit terminations; since this is in essence a Thevenin equivalent circuit, Z_{in} represents an input impedance and E_{nt} a total noise voltage at the signal frequency. By obtaining an expression for the input impedance, the input resistance and the gain of a multiple-idler parametric amplifier can be determined.

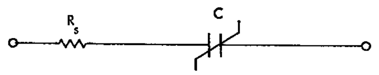

Fig. 10. Simplified equivalent circuit of a varactor diode.

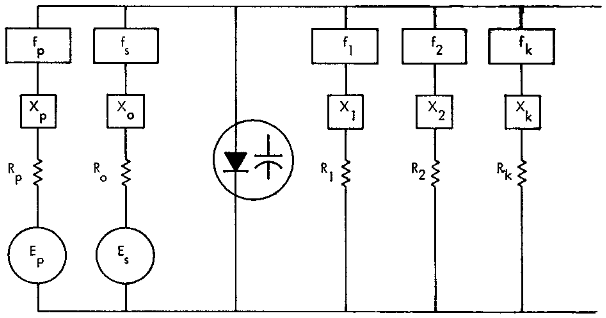

Fig. 11. General form of a multiple-idler parametric amplifier.

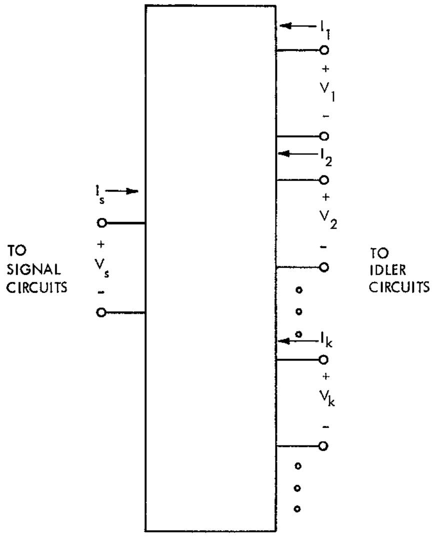

Fig. 12. Repreientation of multiple-idler parametric amplier.

Input Resistance

Because an impedance terminates the varactor diode at every small signal frequency, the various voltages and currents are given by

$$V = -Z_{\text{term}} I + E_{n,\text{term}} \tag{36}$$

where the negative signal results from the fact that current is defined as entering a varactor imbedding network as illustrated in Fig. 12.

Combining (33) and (36) results in the relation

$$O = Z_L I + E_{nL} \tag{37}$$

where

$$Z_L = Z_c + Z_{\text{term}}$$

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[11] Penfield and Rafuse,[3] pp. 79–81.

[12] *Ibid.*, pp. 185–190.

and

$$E_{nL} = E_n - E_{n,\text{term}}.$$

The input impedance of the varactor circuit may be determined by examining the input port when all other ports are terminated. This can be done by examining the equation

$$\mathbf{V}_s = \mathbf{Z}_L\mathbf{I} + \mathbf{E}_{nL} \qquad (Z_{Ts}, E_{nTs} = 0), \tag{38}$$

where $\mathbf{V}_s$ is a column matrix with zero entries everywhere except for the term V_s. The notation that the signal frequency terminal impedance and the associated noise voltage are zero does not imply that this is physically true; it indicates that these values are not considered when evaluating the input characteristics.

For convenience, let the elements of the matrix $\mathbf{Z}_L$ be denoted as indicated by the following equation:

$$\mathbf{Z}_L = \begin{bmatrix} Z_{ss} & Z_{s1} & Z_{s2} & \cdots \\ Z_{1s} & Z_{11} & Z_{12} & \cdots \\ Z_{2s} & Z_{21} & Z_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{39}$$

From (38) the current matrix is found to be given by

$$\mathbf{I} = \mathbf{Z}_L^{-1}\mathbf{V}_s - \mathbf{Z}_L^{-1}\mathbf{E}_{nL} \qquad (Z_{Ts}, E_{nTs} = 0).$$

The expression for the signal current is therefore

$$I_s = \frac{Z_{ss}{}^c}{\Delta_z} V_s - \sum_k \frac{Z_{ks}{}^c}{\Delta_z} E_{nLk} \qquad (Z_{Ts}, E_{nTs} = 0) \tag{40}$$

where Δ_z is the determinant of the matrix $\mathbf{Z}_L$, and $Z_{ks}{}^c$ is the cofactor of the Z_{ks} element in the $\mathbf{Z}_L$ matrix. This expression may also be written as

$$V_s = \frac{\Delta_z}{Z_{ss}{}^c} I_s + \sum_k \frac{Z_{ks}{}^c}{Z_{ss}{}^c} E_{nLk} \qquad (Z_{Ts}, E_{nTs} = 0). \tag{41}$$

Comparing (35) with (41) it is seen that the input impedance is given by

$$Z_{\text{in}} = \frac{\Delta_z}{Z_{ss}{}^c} (Z_{Ts} = 0). \tag{42}$$

By expanding this expression, the input impedance is found to be

$$Z_{\text{in}} = Z_{ss} + Z_{s1}\frac{Z_{s1}{}^c}{Z_{ss}{}^c} + Z_{s2}\frac{Z_{s2}{}^c}{Z_{ss}{}^c} + \cdots (Z_{Ts} = 0). \tag{43}$$

The input resistance is therefore simply Re Z_{in} or

$$R_{\text{in}} = R_s + \text{Re}\left[Z_{s1}\frac{Z_{s1}{}^c}{Z_{ss}{}^c}\right] + \text{Re}\left[Z_{s2}\frac{Z_{s2}{}^c}{Z_{ss}{}^c}\right] + \cdots$$

$$(Z_{Ts} = 0). \tag{44}$$

Noise Temperature

By examining the expression for the signal frequency voltage of a multiple-idler parametric amplifier, (41), it is seen that the total noise voltage is given by

$$E_{nt} = \sum_k \frac{Z_{ks}{}^c}{Z_{ss}{}^c} E_{nLk} \qquad (Z_{Ts}, E_{nTs} = 0). \tag{45}$$

In terms of the sideband noise sources at frequencies $k\omega_p - \omega_s$, this becomes

$$E_{nt} = E_{nLs} + \frac{Z_{1s}{}^c}{Z_{ss}{}^c} E_{nL1} + \cdots \qquad (Z_{Ts}, E_{nTs} = 0). \tag{46}$$

To calculate the amplifier's noise temperature, it is necessary to realize that noise temperature of a negative resistance amplier is given by[13]

$$T = -\frac{\overline{|E_{nt}|^2}}{4R_{\text{in}}K\Delta f}, \text{ where } K = \text{Boltzmann's Constant} \tag{47}$$

and that

$$\overline{|E_{nLk}|^2} = 4KT_kR_k\Delta f + 4KT_dR_s\Delta f \tag{48}$$

where R_k, the external resistance in the kth idler, is set equal to zero where $k = 0$.

By combining the information contained in (44) and (46)–(48) the noise temperature of a multiple-idler parametric amplifier is found to be

$$T = -\frac{R_sT_d + \dfrac{|Z_{1s}{}^c|^2}{|Z_{ss}{}^c|^2}(R_sT_d + R_1T_1) + \dfrac{|Z_{2s}{}^c|^2}{|Z_{ss}{}^c|^2}(R_sT_d + R_2T_2) + \cdots}{R_s + \text{Re}\left[Z_{s1}\dfrac{Z_{s1}{}^c}{Z_{ss}{}^c}\right] + \text{Re}\left[Z_{s2}\dfrac{Z_{s2}{}^c}{Z_{ss}{}^c}\right] + \cdots}. \tag{49}$$

Equation (49) indicates what the noise temperature is for any multiple-idler parametric amplifier; however, it does not indicate the minimum achievable temperature, $T_{\min}$, or how this minimum can be achieved. What is needed is a technique which separates the circuit variables from constants independent of the circuit. Penfield and Rafuse have done this by defining parameters A_k, r_k, θ_k, and r_k' as follows:[14]

$$A_k = \frac{|S_k|^2}{\omega_s{}^2R_s{}^2}\frac{T_dR_s}{T_dR_s + T_kR_k} \qquad (R_0 = 0), \tag{50a}$$

$$r_ke^{j\theta k} = -\frac{Z_{ks}{}^c}{Z_{ss}{}^c}\frac{S_k}{j\omega_sR_s}, \tag{50b}$$

$$r_k' = \frac{A_k}{\sum_k A_k}\left(1 + \sqrt{1 + \sum_k A_k}\right), \tag{50c}$$

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[13] P. Penfield, Jr., "Noise in negative-resistance amplifiers," *IRE Trans. on Circuit Theory*, vol. CT-7, pp. 166–170, June 1960.

[14] Penfield and Rafuse,[3] pp. 191–196.

where r_k and θ_k are the magnitude and phase of the quantity in (50b). Penfield and Rafuse have then shown that the noise temperature becomes

$$T = \frac{2T_d}{\sum_k A_k}\left(1 + \sqrt{1 + \sum_k A_k}\right) + T_d \frac{\sum_k \left| r_k e^{j\theta k} - r_k' \right|^2 / A_k}{\sum_k r_k \cos \theta_k - 1} \cdot \tag{51}$$

Since the first term is independent of $\boldsymbol{Z_c}$, it is the minimum noise temperature which can be achieved. In other words,

$$T_{\min} = \frac{2T_d}{\sum_k A_k}\left(1 + \sqrt{1 + \sum_k A_k}\right). \tag{52}$$

The value of this minimum can be lowered by increasing A_k. This can be done most readily by letting $T_l R_k = 0$. In this case, the noise temperature becomes

$$T_{\min} = T_d \frac{2\omega_s}{m_t \omega_c}\left[\frac{\omega_s}{m_t \omega_c} + \sqrt{1 + \left(\frac{\omega_s}{m_t \omega_c}\right)^2}\right], \tag{53}$$

where m_t is the total modulation ratio defined by

$$m_t = \left[\sum_k \frac{|S_k|^2}{(S_{\max} - S_{\min})^2}\right]^{1/2}. \tag{54}$$

In order to achieve this minimum in a given amplifier circuit, it is necessary that the second term in (51) be made equal to zero. This is possible if and only if the relation

$$r_k e^{j\theta k} = r_k' \tag{55}$$

is satisfied for each and every k. By using this condition, it becomes possible to determine the optimum pump frequency which produces the minimum of (53). However, under certain idler configurations, physically realizable conditions may not exist to achieve this minimum.

APPENDIX II

PROOF OF THEOREM

In order to illustrate the proof of this theorem, the determinant of a specific matrix will be evaluated; the generalization to a matrix of arbitrary order is evident from this. In particular, then, the determinant of the 6 by 6 matrix

$$M = \begin{bmatrix} a_{00} & a_{01} & 0 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & 0 & 0 & 0 \\ 0 & a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix} \tag{56}$$

will be found.

This determinant may be evaluated by converting it to a triangular form and then finding the product of the diagonal elements. The lower triangular form will become

$$|M| = \begin{vmatrix} b_{00} & 0 & 0 & 0 & 0 & 0 \\ a_{10} & b_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{21} & b_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{32} & b_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{43} & b_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{54} & b_{55} \end{vmatrix}. \tag{57}$$

The values of the diagonal elements must now be calculated. Therefore

$$b_{55} = a_{55} \tag{58}$$

$$b_{44} = a_{44} - \frac{a_{45}}{a_{55}} a_{54} = a_{44}\left(1 - \frac{a_{45}a_{54}}{a_{44}a_{55}}\right). \tag{59}$$

The expression $a_{45}a_{54}/a_{44}a_{55}$ is determined by four adjacent elements forming a loop within the matrix. It will, therefore, be designated as a loop quotient and will be indicated as Q_5 since it is the fifth such loop within this matrix. Therefore,

$$b_{44} = a_{44}(1 - Q_5) \tag{60}$$

$$b_{33} = a_{33} - \frac{a_{33}}{b_{44}} a_{43} = a_{33}\left(1 - \frac{Q_4}{1 - Q_5}\right) \tag{61}$$

$$b_{22} = a_{22} - \frac{a_{23}}{b_{33}} a_{32} = a_{22}\left(1 - \cfrac{Q_3}{1 - \cfrac{Q_4}{1 - Q_5}}\right) \tag{62}$$

$$b_{11} = a_{11} - \frac{a_{12}}{b_{22}} a_{21} = a_{11}\left(1 - \cfrac{Q_2}{1 - \cfrac{Q_3}{1 - \cfrac{Q_4}{1 - Q_5}}}\right) \tag{63}$$

$$b_{00} = a_{00} - \frac{a_{01}}{b_{11}} a_{10} = a_{00}\left(1 - \cfrac{Q_1}{1 - \cfrac{Q_2}{1 - \cfrac{Q_3}{1 - \cfrac{Q_4}{1 - Q_5}}}}\right). \tag{64}$$

For notational simplicity, the loop functions will be denoted as B_{mn} where

$$B_{mn} = \left(1 - \cfrac{Q_m}{1 - \cfrac{Q_m + 1}{\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \hline 1 - Q_n \end{array}}}\right). \tag{65}$$

Therefore

$$b_{00} = a_{00}B_{15}; \quad b_{11} = a_{11}B_{25}; \quad b_{22} = a_{22}B_{35}; \quad b_{33} = a_{33}B_{45}; \quad b_{44} = a_{44}B_{55}; \quad b_{55} = a_{55}. \tag{66}$$

Hence the determinant of the matrix is given by

$$|M| = a_{00}a_{11}a_{22}a_{33}a_{44}a_{55}B_{15}B_{25}B_{35}B_{45}B_{55}. \tag{67}$$

In order to avoid indeterminant forms, it is necessary that none of the main diagonal terms of the original matrix be zero.

This derivation may be easily extended to any order matrix to give

$$|M| = a_{00}a_{11}a_{22} \cdots a_{nn}B_{1n}B_{2n} \cdots B_{nn} \tag{68}$$

and hence the given theorem is proved.

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A Simple Diode Parametric Amplifier Design for Use at S, C, and X Band

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Abstract—**A simple diode parametric amplifier is described which has been designed for use at *S*, *C*, and *X* band frequencies. Bandwidth and noise measurements show the performance to be substantially in agreement with the theoretical predictions. Details are given of compensating circuits which improve the gain-frequency response of the amplifier by up to 5.6 times.**

I. INTRODUCTION

MANY PARAMETRIC amplifier designs involve the use of distributed signal and idling circuits with consequent reduction of the gain bandwidth performance which is available from the amplifier. The techniques described in this paper show that lumped techniques can be utilized to give improved bandwidths, and that reactance compensating networks can be used to increase the gain-frequency response even further.

A. General Principles of Parametric Amplifier Circuit Design

A satisfactory diode parametric amplifier design must fulfill the following requirements:

a) The diode must be coupled to a circuit which provides the appropriate resonating reactance and appropriately overcoupled source resistance at the signal frequency. The diode must also be coupled to a reactance (preferably lossless) which provides a conjugate reactance at a convenient idling frequency. The noise figure F of a negative-resistance amplifier can be shown [1] to be given by the expression

$$F = 1 + \frac{R_d}{R_g} + A\frac{\omega_1}{\omega_2}\frac{(R_d + R_g)}{R_g} \tag{1}$$

where R_d is the diode spreading resistance, R_g the source resistance, A the ratio of negative resistance in the signal circuit to total positive resistance in the signal circuit, ω_1 the signal frequency, and ω_2 the idling frequency. R_g is selected so that the diode current due to pumping is small, typically one microampere, in order that shot noise associated with this current shall remain negligible. There is no contribution to (1) from the circulator since it is assumed to have infinite isolation and zero insertion loss.

b) Signal, idling, and pump energies must be confined to the appropriate regions of the circuit in order that maximum operating bandwidth can be obtained and in order that minimum pump power will be required to operate the amplifier.

c) Both signal and idling circuits should be lumped (small compared with the wavelength) or, if distributed, have an electrical length less than one-tenth of the wavelength in order that the Q of the circuit shall not be increased by the distributed nature of the reactance. The available gain-bandwidth product ($G_{av}^{1/2}B$) of a negative resistance amplifier operated in conjunction with a circulator is given by the

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